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Properties Concerning Dense Subsets of Sorgenfrey Spaces and Michael Spaces

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Abstract

In this paper, we investigate the linearly ordered topological spaces and their Sorgenfrey spaces, and study four properties concerning dense subspaces of such spaces that were introduced by H. R. Bennett, D. J. Lutzer and S. D. Purisch. As a result, we show that if a linearly ordered space has Property IV, then so does its Sorgenfrey space. In the fourth section, we prove theorems concerning such properties and the weak perfectness of the Michael spaces.

Key words : Properties I, II, III and IV, linearly ordered space, Sorgenfrey space, Michael space, weak perfectness.

1. Definitions of the four properties concerning dense subsets

The following definitions were given in [BLP]. (See also [H2]). For a generalized ordered space (GO-space) X that has one of the properties in the definitions, we have a fact : the density of X = the cellularity of X. A GO-space is defined as a subspace of a linearly ordered topological space. (See Section 2).

Definition 1. A topological space X is said to have *Property I* if and only if there exists a σ-closed discrete dense subset D of X, that is, $D = \cup\{D(n) : n \in N\}$ is a dense subset of X such that $D(n)$ is a closed discrete subset of X for every $n \in N$. N denotes the set of natural numbers.

Definition 2. A space X is said to have *Property II* if and only if there is a dense metrizable subspace of X.

Definition 3. A space X is said to have *Property III* if and only if, for each $n \in N$, there are an open subset $U(n)$ of X and a relatively closed discrete subset $D(n)$ of $U(n)$ such that, for a point p and an open subset G of X that contains p, there exists an $n \in N$ such that $p \in U(n)$ and $G \cap D(n)$

$\neq \phi$. (See also [BL], [H1]).

Definition 4. A space X is said to have *Property IV* if and only if there exists a σ-relatively discrete dense subset D of X, that is, $D = \cup\{D(n) : n \in N\}$ is a dense subset of X such that $D(n)$ is a relatively discrete subspace of X for every $n \in N$. "Relatively discrete" means "discrete as a subspace".

2. Linearly ordered spaces and Sorgenfrey spaces

Let $(X, <)$ be a linearly ordered set. We can consider two topologies on $(X, <)$. One of them is a linearly ordered topological space (LOTS) and the other is a Sorgenfrey space. A LOTS $(X, <, I)$ has the order topology defined by <, that is, a basic open neighborhood of x in the LOTS is of the form $]y, z[$, where $y < x < z$. The order topology is often called the interval topology. That is the reason why the letter I is used. A basic open neighborhood of x in a Sorgenfrey space $(X, <, S)$ is of the form $[x, y[$, where $x < y$. We usually abbreviate $(X, <, I)$ as (X, I), and $(X, <, S)$ as (X, S). We also write $(X, <)$ as X.

It is interesting to consider the relationship between (X, I) and (X, S), where X is a linearly ordered set. In this paper, we investigate whether the following statement is true: if (X, I) has Property P, then so does (X, S) and vice versa, where P is one of the four properties I, II, III, and IV defined in the first section.

The following two lemmas are necessary to prove Proposition 1.

Lemma 1. *Let X be a linearly ordered set and (X, S) the Sorgenfrey space. Let $D = \{x \in X : \{x\}$ is open in $(X, S)\}$. Then there exists a subset S of $X - D$ such that $S = \cup\{S_\alpha : \alpha \in A\}$, where*

(1) *S_α is a convex, open subset of (X, S) for every $\alpha \in A$. (S_α is said to be convex, if $x < y < z$ with $x, z \in S_\alpha$, then $y \in S_\alpha$).*

(2) *$S_\alpha \cap S_\beta = \phi$ for $\alpha \neq \beta$.*

(3) *$S \cup D$ is a dense subset of (X, S).*

Proof. If D is a dense subset of (X, S), then we set $S = \phi$. Suppose that D is not dense in (X, S). Let $x \notin D$. Then, the following two cases occur.

Case 1. Any neighborhood of x contains a point of D.

Case 2. There exists $y \in X$ such that $x < y$, $[x, y[\neq \phi$ and $[x, y[\cap D = \phi$.

Consider Case 2 first to construct $S_\alpha = [x_\alpha, y_\alpha[$, we shall consider the following four cases: (i) if $]\leftarrow, x[\cap D = \phi$, then let $]x_\alpha, x[=]\leftarrow, x[$. (ii) if there exists $\max(]\leftarrow, x[\cap D)$, then we set x_α = the successor of $\max(]\leftarrow, x[\cap D)$. (iii) if there exists $\sup(]\leftarrow, x[\cap D)$ in $X - D$, then we set $x_\alpha = \sup(]\leftarrow, x[\cap D)$. (iv) if $\sup(]\leftarrow, x[\cap D)$ is a gap, then we set x_α = the gap. Next we explain how to choose y_α: let $y \notin D$, $x_\alpha < y$ and $]x_\alpha, y[\cap D = \phi$.

(i) if $[y, \rightarrow[\cap D = \phi$, then let $[y, y_\alpha[= [y, \rightarrow[$. (ii) if there exists $\min([y, \rightarrow[\cap D)$, then let $y_\alpha = \min([y, \rightarrow[\cap D)$. (iii) if there exists $\inf([y, \rightarrow[\cap D)$ in $X - D$, then let $y_\alpha = \inf([y, \rightarrow[\cap D)$. (iv) if $\inf([y, \rightarrow[\cap D)$ is a gap, then we set y_α = the gap. Now, let $S_\alpha = [x_\alpha, y_\alpha[$ and $S = \cup\{S_\alpha : \alpha \in A\}$. Then we can easily show (1) and (2) stated in the lemma. To prove (3), $x \in X - (S \cup D)$. Note that $x \neq x_\alpha$. If $x = y_\alpha$, there are infinitely many points of $[y_\alpha, \rightarrow[\cap D$. So any neighborhood of y_α meets D. If x is distinct from x_α, y_α, then by Case 1 above, any neighborhood of x contains a point of D. This shows (3) and completes the proof of Lemma 1.

Lemma 2. *Let D be a countable subset of a Sorgenfrey space $(X, \mathcal{S})$. Then D is metrizable.*

Proof. Let $D = \{d_n : n \in N\}$ be a countable subset of X. Let $\mathcal{B}(n, m) = \{[d_n, d_m[\}$, where $d_n < d_m$. It is easy to see that $\mathcal{B} = \cup\{\mathcal{B}(n, m) : (n, m) \in N \times N\}$ is a base for $(D, \mathcal{S}|D)$. Each family $\mathcal{B}(n, m)$ consists of a single element, so it is discrete. Hence $\mathcal{B}$ is a σ-discrete base for the space $(D, \mathcal{S}|D)$. By the Bing's Metrization Theorem, $(D, \mathcal{S}|D)$ is metrizable. This completes the proof.

The following are results concerning Properties II and IV for a Sorgenfrey space, although Proposition 1 is a partial result. On the contrary, it is unknown if we have a similar result for Property III. See Examples 1 and 2 in the third section for Property I.

Proposition 1. *Let $(X, \mathcal{S})$ be a Sorgenfrey space. If every $(S_\alpha, \mathcal{S} | S_\alpha)$ is separable, then $(X, \mathcal{S})$ has Property II, where S_α's are defined in Lemma 1 and $\mathcal{S}|S_\alpha$ denotes the topology on S_α induced by $\mathcal{S}$.*

Proof. Since $(S_\alpha, \mathcal{S}|S_\alpha)$ is separable, there exists a countable dense subset T_α of $(S_\alpha, \mathcal{S}|S_\alpha)$ for every $\alpha \in A$. Hence $(T_\alpha, \mathcal{S}|T_\alpha)$ is metrizable for every $\alpha \in A$ by Lemma 2. It follows from Lemma 1 that $\{T_\alpha : \alpha \in A\}$ and D are mutually disjoint. Set $T = \cup\{T_\alpha : \alpha \in A\}$. Since $T_\alpha \subset]x_\alpha, y_\alpha[$ for every $\alpha \in A$, it is easy to see that $T \cup D$ is a topological sum of $\{T_\alpha : \alpha \in A\}$ and D. Hence $T \cup D$ is metrizable. Since $T \cup D$ is dense in $(S \cup D, \mathcal{S}|S \cup D)$ and $S \cup D$ is dense in $(X, \mathcal{S})$, $T \cup D$ is a dense subset of $(X, \mathcal{S})$. Therefore, $(X, \mathcal{S})$ has Property II. This completes the proof.

Remark 1. Let R be the set of real numbers. Then $D = \{x \in R : \{x\}$ is open in $(R, \mathcal{S})\}$ is empty. Hence $S = R$ by Lemma 1. Since the Sorgenfrey line $(R, \mathcal{S})$ is separable, it has a dense metrizable subspace $(Q, \mathcal{S}|Q)$ by Lemma 2, where Q is the set of rational numbers. This justifies the assumption of separability in Proposition 1.

Theorem 1. *If $(X, \mathcal{I})$ has Property IV, then so does $(X, \mathcal{S})$.*

Proof. Let $D = \cup\{D(n) : n \in N\}$ be a σ-relatively discrete dense subset of $(X, \mathcal{I})$. Let

$$D(0) = \{x \in X - D :]a, x[\neq \phi \text{ for any } a < x, \text{ and there exists } y > x \text{ such that }]x, y[= \phi\},$$

and $D' = D \cup D(0)$. Then D' is a dense subset of $(X, \mathcal{I})$. To prove this, let x be a point of $X - D'$, and $[x, y[$ a neighborhood of x in $(X, \mathcal{S})$. If $]x, y[$ is not empty, then $]x, y[\cap D \neq \phi$. Hence $[x, y[\cap$

$D' \neq \phi$. If $]x, y[$ is empty, then there exists $a < x$ such that $]a, y[= \{x\}$, because if $]a, x[\neq \phi$ for any $a < x$, then $x \in D(0)$. Hence $x \in D'$. This contradicts the assumption $x \notin D'$. Hence $\{x\}$ is open in (X, I) and $x \in D$. Since x does not belong to D', this case does not occur. It is clear that $D(n)$, $n \geqq 1$, is relatively discrete in (X, S). Since for each $x \in D(0)$, $\{x\}$ is open in (X, S), $D(0)$ is relatively discrete in (X, S). Hence D' is a σ-relatively discrete dense subset of (X, S). Therefore, (X, S) has Property IV. This completes the proof.

3. Miscellaneous counterexamples

Example 1. Even if (X, I) has Property I, (X, S) does not necessarily have Property I:
We consider a lexicographically ordered set $X = [0, 1] \times \{0, 1\}$. Since (X, I) is separable, it has Property I. But, (X, S) has $[0, 1] \times \{0\}$ as a dense discrete subset. Hence it is not perfect (see Definition 5 in Section 4). Since a GO-space with Property I is perfect, it does not have Property I.

Example 2. Even if (X, S) has Property I, (X, I) does not necessarily have Property I:
We consider $X = \omega_1$, the set of countable ordinals. It is clear that (X, S) is discrete, hence it has Property I. Since (X, I) is not paracompact, it does not have Property I.

Example 3. Even if (X, S) has Property II, (X, I) does not necessarily have Property II:
Let $X = I^{\omega_1} \times \{0, 1\}$ be a lexicographically ordered set, where I^{ω_1} itself is ordered lexicographically. Then (X, S) has Property II. It is clear that $I^{\omega_1} \times \{0\}$ is a dense, discrete subset of (X, S). Hence $I^{\omega_1} \times \{0\}$ is a dense metrizable subspace of (X, S). To prove that (X, I) does not have Property II, it is sufficient to show that (X, I) is not first countable at any point. As was proved in [BLP], for a GO-space X, if X has a dense metrizable subspace D, then X is first countable at each point of D. Let $s = (s_1, s_2, \cdots, s_\alpha, \cdots; 0) \in I^{\omega_1} \times \{0\}$. If there exists an increasing seqence $\{s_n : n \in N\}$ in (X, I) that converges to s, then there exist countably many points of $I^{\omega_1} \times \{0\}$ that converge to s. As is well known, I^{ω_1} is not first countable at any point. This is a contradiction.

Example 4. There is another example to provide the same situation as Example 3:
Let X be a linearly ordered topological space (LOTS) defined in [BL], Example 5.5. X has Property III, but it is not first-countable at any point of X. Precisely,

$$X = \{(\alpha_1, \cdots, \alpha_n, \omega_1, \omega_1, \omega_1, \cdots) \in [0, \omega_1]^\omega : \alpha_i < \omega_1, i = 1, 2, \cdots, n, n \geqq 0\},$$

where if $n = 0$, then $(\overline{\omega_1}) = (\omega_1, \omega_1, \omega_1, \cdots)$ is a single point. Let

$$X(n) = \{x \in X : \text{the length of } x \text{ is } n\},$$

where the length of $x = (\alpha_1, \cdots, \alpha_n, \omega_1, \omega_1, \omega_1, \cdots)$ is n if $\alpha_n < \omega_1$. Then $X = \cup\{X(n) : n \geqq 0\}$. Since each $X(n)$ is relatively discrete in (X, I) as shown in [BL]. Hence $X(n)$ is also relatively discrete in (X, S). We show that every $X(n)$ is a closed subset of (X, S). Let $x \in X - X(n)$. If $n = 0$, it is easy to prove that. So let $n > 0$. Let $x \in X(i)$, $0 < i < n$. If $x = (\alpha_1, \cdots, \alpha_i, \omega_1, \cdots \omega_1, \omega_1, \overline{\omega_1})$, then let $y = (\alpha_1, \cdots, \alpha_i + 1, 0, \cdots, 0, \omega_1, \overline{\omega_1})$. It is easy to show that $[x, y[\cap X(n) = \phi$. Next, let $x \in X(i)$, $i > n$. For $x = (\alpha_1, \cdots, \alpha_i, \omega_1, \omega_1, \omega_1, \cdots)$, let $y = (\alpha_1, \cdots, \alpha_i + 1, \omega_1, \omega_1, \omega_1, \cdots)$. Then $[x, y[\cap X(n) =$

ϕ. Hence $X(n)$ is a closed discrete subset of (X, S). Therefore (X, S) has Property I. Hence it has Property II. Since (X, I) is not first-countable at any point, it does not have Property II as proved in [BLP].

Remark 2. $X(n)$ in Example 4 is not necessarily a closed subset in (X, I). For example, consider $X(2)$ and a point $x = (\omega, \overline{\omega_1})$ that does not belong to $X(2)$. We can show that for any $y \in X$, $y < x$, $]y, x] \cap X(2) \neq \phi$. Let $y = (\alpha_1, \cdots, \alpha_i, \overline{\omega_1})$ be less than x. Case 1: let $\alpha_1 = \omega$. Then, $\alpha_2 < \omega_1$. To see this, suppose that $\alpha_2 = \omega_1$. Then for all $i > 1$, $\alpha_i = \omega_1$. Hence $y = x$. This contradicts $y < x$. Let $z = (\alpha_1, \alpha_2 + 1, \overline{\omega_1})$. Then $z \in X(2)$ and $y < z < x$. Hence $]y, x] \cap X(2) \neq \phi$. Case 2: let $\alpha_1 < \omega$. Then $\alpha_1 = n \in N$. Let $z = (\alpha_1 + 1, \alpha_2, \overline{\omega_1}) \in X(2)$ and $y < z < x$. Hence $]y, x] \cap X(2) \neq \phi$. Case 3: the case $\alpha_1 > \omega$ does not occur because $y < x$. This completes the proof.

Example 5. Even if (X, S) has Property III, (X, I) does not necessarily have Property III: Let $X = \omega_1$. Then (X, S) is discrete. Hence it has Propety III. Because (X, I) is not paracompact, it does not have Property III by Proposition 4.2 in [BL].

Example 6. Even if (X, S) has Property IV, (X, I) does not necessarily have Property IV: Let $X = I^{\omega_1} \times \{0, 1\}$ be a lexicographically ordered set, where I^{ω_1} itself has the lexicographic order. It is clear that a subspace $S = I^{\omega_1} \times \{0\} \cup \{(1, 1, \cdots, 1; 1)\}$ is open, relatively discrete in (X, S). We show that S is a dense subset of (X, S). Let

$$(y, 1) = (y_1, \cdots, y_\alpha, \cdots; 1) \text{ and } (z, 0) = (z_1, \cdots, z_\alpha, \cdots; 0)$$

be points of $I^{\omega_1} \times \{0, 1\}$ that $(y, 1) < (z, 0)$. Since $y < z$, there exists $\alpha < \omega_1$ such that $y_\alpha < z_\alpha$ and $y_\beta = z_\beta$ for $\beta < \alpha$. Let $(u, 0) = (y_1, \cdots, u_\alpha, \cdots; 0)$, where $y_\alpha < u_\alpha < z_\alpha$. Hence $(y, 1) < (u, 0) < (z, 0)$. Therefore, $(u, 0) \in [(y, 1), (z, 0)[$. Hence (X, S) has Property II, and hence (X, S) has Property IV. Suppose that (X, I) has Property IV. Since (X, I) is a compact LOTS, it has Property II by Corollary 4.7 in [BLP]. Then (X, I) is first-countable on a dense subset D. But, neither $I^{\omega_1} \times \{0\}$ nor $I^{\omega_1} \times \{1\}$ is first-countable at any point. Hence (X, I) does not have Property IV.

4. Four properties and the weak perfectness of Michael spaces

Let P be a subset of the unit interval $X = [0, 1]$. We topologize X as follows: for each point $p \in P$, $\{p\}$ is open and for each point $x \in X - P$, we agree to endow the usual Euclidean neighborhoods. This space is written $M(P)$ and is said to be a Michael space. The letter M stands for the topology of $M(P)$, and E is used for the usual Euclidean topology on $[0, 1]$ through the remaining of the paper.

Theorem 2. *Any Michael space $M(P)$ has Properties II, III and IV.*

Proof. About Property II: if P is dense in $X = [0, 1]$, then it is easy to see that $M(P)$ has

Property II. Suppose that P is not a dense subset of X. Then, by Lemma 1, we can write $X-P=\cup\{S_\alpha:\alpha\in A\}$, where S_α is an open convex subset of $[0,1]$ for every $\alpha\in A$. Since $[0,1]$ is hereditarily separable, there exists a countable dense subset $D=\{d_n:n\in N\}$ of $X-P$. Let $B(0)=\{\{p\}:p\in P\}$, and $B(n,m)=\{]d_n-(1/m),d_n+(1/m)[\}$ for each $(n,m)\in N\times N$. Then we can easily show that

$$B=B(0)\cup(\cup\{B(n,m):(n,m)\in N\times N\})$$

is a σ-discrete base for $D\cup P$. Hence $D\cup P$ is a dense metrizable subspace of $M(P)$ by the Bing's Metrization Theorem. Hence $M(P)$ has Property II. About Property III: it is sufficient to show that $M(P)$ is quasi-developable by Lemma 3.4 and Proposition 1.6 in [BL]. In fact, B is a quasi-development. To show that, let x be a point of an open set U, where $x\notin D\cup P$. Then there exists $(n,m)\in N\times N$ such that $x\in]d_n-(1/m),d_n+(1/m)[\subset U$. If $x\in D\cup P$, then it is easy to get a similar consequense. Since the quasi developability means the existence of a σ-minimal base, $M(P)$ has Property III by [BL] (see also [H]). It is shown in [BLP] that Property II (or III) implies Property IV. This completes the proof of Theorem 2.

The notion of weak perfectness was introduced by L. J. Kočinac [K]. Before the discussion, we give the definition and a theorem that are needed to prove our result. Theorems 3 and 5 were announced in [BHL], but an explicit proof was not given in the paper, so we give a direct proof for the sake of convenience.

Definition 5. A topological space X is called weakly perfect if and only if, for every closed subset C of X, there exists a dense subset D of C such that D is a G_δ-subset of X. If we can take $D=C$, then X is said to be perfect.

Theorem 3. *Suppose that $M(P)$ is weakly perfect. Then, for any E-closed subset C of $[0,1]$ such that $C-P$ is E-dense in C, $P\cap C$ is a first category subset of C. (The converse is also true).*

Proof. Let C be an E-closed subset of $[0,1]$ such that $C-P$ is E-dense in C. Let $C^*=C-P$. Then C^* is an M-closed subset of $M(P)$. Since $M(P)$ is weakly perfect, there exists an M-dense subset S of C^* that is an M-G_δ-subset of $M(P)$, say, $S=\cap\{G(n):n\in N\}$, where $G(n)$ is an E-open subset of $M(P)$ for every $n\in N$. Let

$$H(n)=\{x\in C:\text{there exist } s,t\in R \text{ such that } s<x<t \text{ and }]s,t[\cap[0,1]\subset G(n)\}.$$

Then, it is easy to see that $H(n)\subset G(n)$ and that $H(n)$ is a relatively E-open subset of C. We show that $H(n)$ is E-dense in C. Let $]a,b[\cap C$ be a non-empty E-open subset of C. Since $C-P$ is E-dense in C, we have $]a,b[\cap(C-P)\neq\phi$. Since S is M-dense in $C^*=C-P$, $]a,b[\cap(C-P)\cap S\neq\phi$. Take a point $y\in]a,b[\cap(C-P)\cap S$. Hence $y\in S\subset G(n)$ for every $n\in N$. Note that $G(n)$ is M-open in $M(P)$. Since $y\notin P$, there exist $y_1,y_2\in R$ such that $y_1<y<y_2$ and $]y_1,y_2[\cap[0,1]\subset G(n)$. Since $y\in C$, $y\in H(n)$. Hence $]a,b[\cap C\cap H(n)\neq\phi$. That shows that $H(n)$ is E-dense in C. Since $\cap\{H(n):n\in N\}\subset\cap\{G(n):n\in N\}=S\subset C-P$, it follows that

$$P\cap C\subset C-[\cap\{H(n):n\in N\}]=\cup\{(C-H(n)):n\in N\}.$$

It is clear that $C-H(n)$ is relatively E-closed in C and nowhere dense in C. Therefore, $P\cap C$ is a first category subset of C. The proof of the converse statement will be given in Theorem 5, below.

In the following theorem, let P denote the set of irrational numbers in $[0, 1]$. Hence $M(P)$ is the usual Michael line.

Theorem 4. *Let P be the the set of irrational numbers in $[0, 1]$. Then $M(P)$ is not weakly perfect.*

Proof. Suppose that $M(P)$ is weakly perfect. Take a closed set $C=[0, 1]$. Then $C-P$ is dense in C, because P is the irrational numbers in $[0, 1]$. Hence, by Theorem 3, $P\cap C$ is a first category subset of C. This means P is a first category subset of $[0, 1]$. This contradicts a well known fact.

The following is the converse to Theorem 3.

Theorem 5. *Suppose that for any E-closed subset of $[0, 1]$ such that $C-P\subset C$ is E-dense in C, $P\cap C$ is a first category subset of C. Then $M(P)$ is weakly perfect.*

Proof. Let C be a M-closed subset of $M(P)$. Let $K_1=\mathrm{Cl}(C-P, [0, 1])$, the E-closure of $C-P$ in $[0, 1]$. Then $K_1-(C-P)\subset P$. To show this, let x be a point of $K_1-(C-P)$. Suppose that $x\notin P$. Since $x\in K_1$, $V(x)\cap(C-P)\neq\phi$ for any E-neighborhood $V(x)$. Hence $V(x)\cap C\neq\phi$. Since C is M-closed in $M(P)$ and $x\notin P$, $x\in \mathrm{Cl}(C, M(P))=C$. Hence $x\in C-P$. This contradicts that $x\in K_1-(C-P)$. Hence we can write $K_1=(C-P)\cup P'$, where P' is a subset of P. To see that $K_1-P\subset K_1$ is E-dense, let $x\in K_1$ and $V(x)$ a neighborhood of x in $[0, 1]$. Then $V(x)\cap(C-P)\neq\phi$. Since $C-P\subset K_1-P$, $V(x)\cap(K_1-P)\neq\phi$. Hence K_1-P is E-dense in K_1. By the assumption, $P\cap K_1$ is a first category subset of K_1. Hence $P\cap K_1$ is contained in $\cup\{F(n) : n\in N\}$, where $F(n)$ is E-closed in K_1 and nowhere dense in K_1. Since $K_1-F(n)$ is relatively E-open in K_1 and E-dense in K_1, by the Baire Category Theorem, $\cup\{K_1-F(n) : n\in N\}$ is E-dense in K_1, because K_1 is compact in $[0, 1]$. Note that $(P\cap K_1)\cap(K_1-\cup\{F(n) : n\in N\})=\phi$. Therefore,

$$(P\cap K_1)\cap(\cap\{K_1-F(n) : n\in N\})=\phi.$$

That means $\cap\{K_1-F(n) : n\in N\}$ does not contain any point of P. Let

$$D=(P\cap C)\cup[\cap\{K_1-F(n) : n\in N\}].$$

Then $D\subset C$. To see this, let $x\in\cap\{K_1-F(n) : n\in N\}\subset K_1$. Since $x\notin P$, any M-neighborhood of x is a Euclidean. Hence for any neighborhood $V(x)$ of x, we have $V(x)\cap(C-P)=\phi$, and hence $V(x)\cap C\neq\phi$. Therefore, $x\in\mathrm{Cl}(C, M(P))=C$, where $\mathrm{Cl}(C, M(P))$ denotes the M-closure of C in $M(P)$. We now show that $D=(P\cap C)\cup[\cap(K_1-\cup\{F(n) : n\in N\}]$ is M-dense in C. To see this, let $x\in C-D$. Note that $x\notin P$. Let $V(x)$ be an E-neighborhood of x. Since $x\in K_1$ and $\cap\{K_1-F(n): n\in N\}$ is E-dense in K_1, $V(x)\cap(\cap\{K_1-F(n) : n\in N\})=\phi$. Hence $V(x)\cap D=\phi$. This shows that D is M-dense in C. We show next that D is an M-G_δ-subset of $M(P)$. Since K_1 is E-closed in $[0, 1]$, we may write $K_1=\cap\{W(m) : m\in N\}$, where $W(m)$ is E-open in $[0, 1]$ for every $m\in N$. Since $K_1-F(n)$ is E-open in K_1, we may write $K_1-F(n)=U(n)\cap K_1$, where $U(n)$ is E-open in $[0,$

1] for every $n \in N$. Hence

$$K_1 - F(n) = U(n) \cap [\cap\{W(m) : m \in N\}] = \cap\{U(n) \cap W(m) : m \in N\}.$$

Hence

$$D = (P \cap C) \cup [\cap\{K_1 - F(n) : n \in N\}] = (P \cap C) \cup [\cap\cap\{U(n) \cap W(m) : n \in N, m \in N\}].$$

Since $U(n) \cap W(m)$ is E-open in $[0,1]$, it is also M-open in $M(P)$. Since $P \cap C$ is clearly M-open in $M(P)$, so is $(P \cap C) \cup (U(n) \cap W(m))$. Hence

$$D = \cap\cap\{(P \cap C) \cup (U(n) \cap W(m))) : n \in N, m \in N\}$$

is an M-G_δ-subset of $M(P)$. This completes the proof.

Remark 3. We do not need to assume that C is dense-in-itself in Theorems 3 and 5, although it was assumed in [BHL]. Since $C - P$ is E-dense in C, no points of $P \cap C$ are E-isolated. So the reflections to E-isolated points are not needed when we consider $P \cap C$.

5. Questions

The author does not know the answer to the following question:

Question. *Let P be one of Properties II and III, and suppose that* $(X, \mathcal{I})$ *has P. Does* $(X, \mathcal{S})$ *have P*?

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